Lax Pairs and Darboux Transformations for Euler Equations

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Abstract

In this article, we will report the recent developments on Lax pairs and Darboux transformations for Euler equations of inviscid fluids.

Contents

1 Introduction

The governing equations for the incompressible viscous fluid flow are the Navier-Stokes equations. Turbulence occurs in the regime of high Reynolds number. By formally setting the Reynolds number equal to infinity, the Navier-Stokes equations reduce to the Euler equations of incompressible inviscid fluid flow. One may view the Navier-Stokes equations with large Reynolds number as a singular perturbation of the Euler equations.

Results of T. Kato show that 2D Navier-Stokes equations are globally well-posed in $C^0([0,\infty); H^s(R^2))$, $s > 2$, and for any $0 < T < \infty$, the mild solutions of the 2D Navier-Stokes equations approach those of the 2D Euler equations in $C^0([0,T]; H^s(R^2))$ [10]. 3D Navier-Stokes equations are locally well-posed in $C^0([0,\tau]; H^s(R^3))$, $s > 5/2$, and the mild solutions of the 3D Navier-Stokes equations approach those of the 3D Euler equations in $C^0([0,\tau]; H^s(R^3))$, where τ depends on the norms of the initial data and the external force [8] [9]. Extensive studies on the inviscid limit have been carried by J. Wu et al. [17] [6] [18] [4]. There is no doubt that mathematical study on Navier-Stokes (Euler) equations is one of the most important mathematical problems. In fact, Clay Mathematics Institute has posted the global well-posedness of 3D Navier-Stokes equations as one of the one million dollars problems.

V. Arnold [1] realized that 2D Euler equations are a Hamiltonian system. Extensive studies on the symplectic structures of 2D Euler equations have been carried by J. Marsden, T. Ratiu et al. [14]. S. Friedlander and M. Vishik [7] [16] found a Lax pair for Euler equations written in the Lagrangian coordinates.

Recently, the author [12] [13] found a Lax pair for 2D Euler equations written in the Eulerian coordinates. S. Childress [5] found a Lax pair for 3D Euler equations. In this article, a Darboux transformation for 2D Euler equations and their Lax pair is found. Darboux transformations are powerful tools in generating explicit representations for figure eight

structures in the phase spaces of Hamiltonian partial differential equations [11]. Of course, Darboux transformations were traditionally used for generating multi-soliton solutions to soliton equations [15].

Understanding the structures of solutions to Euler equations is of fundamental interest. Of particular interest is the question whether or not 3D Euler equations have finite time blow up solutions. T. Beale, T. Kato, and A. Majda derived a necessary condition [2].

Our hope is that the Lax pair and the Darboux transformation can be useful in investigating finite time blow up solutions of 3D Euler equations and in establishing global well-posedness of 3D Navier-Stokes equations.

The rest of this article is organized as follows: In section 2, we will present a formulation of Euler equations. In section 3, we will present a recent result on a Lax pair for 2D Euler equations. In section 4, we will present a recent result on a Lax pair for 3D Euler equations by Steve Childress. In section 5, we will present a recent result on a Darboux transformation for 2D Euler equations and their Lax pair. In section 6, a short conclusion is presented.

2 Formulations of Euler Equtions

The three-dimensional incompressible Euler equation can be written in vorticity form,

$$\partial_t \Omega + (u \cdot \nabla)\Omega - (\Omega \cdot \nabla)u = 0 \ , \tag{2.1}$$

where $u = (u_1, u_2, u_3)$ is the velocity, $\Omega = (\Omega_1, \Omega_2, \Omega_3)$ is the vorticity, $\nabla = (\partial_x, \partial_y, \partial_z)$, $\Omega = \nabla \times u$, and $\nabla \cdot u = 0$. u can be represented by Ω for example through Biot-Savart law.

When the fluid flow is of two dimensional, i.e.

$$u = (u_1(t, x, y), u_2(t, x, y), 0) \ ,$$

the vorticity Ω is of the form $\Omega = (0, 0, \Omega_3(t, x, y))$. Dropping the subscript 3 of Ω_3, the 2D Euler equation can be written in the form,

$$\partial_t \Omega + \{\Psi, \Omega\} = 0 \ , \tag{2.2}$$

where the bracket $\{\ ,\ \}$ is defined as

$$\{f, g\} = (\partial_x f)(\partial_y g) - (\partial_y f)(\partial_x g) \ ,$$

where Ψ is the stream function given by,

$$u = -\partial_y \Psi \ , \quad v = \partial_x \Psi \ ,$$

and the relation between vorticity Ω and stream function Ψ is,

$$\Omega = \partial_x v - \partial_y u = \Delta \Psi \ .$$

3 A Lax Pair for 2D Euler Equation

Theorem 3.1 (Li, [12]) *The Lax pair of the 2D Euler equation (2.2) is given as*

$$\begin{cases} L\varphi = \lambda\varphi \ , \\ \partial_t \varphi + A\varphi = 0 \ , \end{cases} \tag{3.1}$$

where

$$L\varphi = \{\Omega, \varphi\} \ , \quad A\varphi = \{\Psi, \varphi\} \ ,$$

and λ is a complex constant, and φ is a complex-valued function.

This theorem was announced in [12] without proof. Here we furnish a detailed proof.

Proof: In order to solve the over-determined system (3.1), one needs a compatibility condition. To derive a compatibility condition, we first take a ∂_t to the first equation,

$$(\partial_t L)\varphi + L(\partial_t \varphi) = \lambda(\partial_t \varphi) = -\lambda A\varphi = -A(\lambda\varphi) = -AL\varphi \ ,$$

i.e.

$$(\partial_t L)\varphi - LA\varphi = -AL\varphi \ ,$$

and finally

$$\Big[(\partial_t L) + [A, L]\Big]\varphi = 0 \ ,$$

where $[A, L] = AL - LA$ is the commutator.

$$(\partial_t L)\varphi = \{(\partial_t \Omega), \varphi\} \ ,$$

$$\begin{aligned} [A, L]\varphi &= \{\Psi, \{\Omega, \varphi\}\} - \{\Omega, \{\Psi, \varphi\}\} \\ &= \Big[\{\Psi, \{\Omega, \varphi\}\} + \{\Omega, \{\varphi, \Psi\}\} \\ &\quad + \{\varphi, \{\Psi, \Omega\}\}\Big] - \{\varphi, \{\Psi, \Omega\}\} \ , \end{aligned}$$

where by Jacobi identity, $[\ \cdot\ ] = 0$. Thus

$$[A, L]\varphi = \{\{\Psi, \Omega\}, \varphi\} \ .$$

Finally

$$\Big[(\partial_t L) + [A, L]\Big]\varphi = \{\partial_t \Omega + \{\Psi, \Omega\}, \varphi\} = 0 \ .$$

Therefore, the 2D Euler equation

$$\partial_t \Omega + \{\Psi, \Omega\} = 0$$

is a compatibility condition. $\square$

Remark 3.1 *A Darboux transformation for the above Lax pair will be presented in section 5. Recently, Darboux transformations have been used in constructing explicit representations of homoclinic structures [11].*

4 A Lax Pair for 3D Euler Equation

Theorem 4.1 (Childress, [5]) *The Lax pair of the 3D Euler equation (2.1) is given as*

$$\begin{cases} L\varphi = \lambda\varphi \ , \\ \partial_t \varphi + A\varphi = 0 \ , \end{cases} \tag{4.1}$$

where

$$L\varphi = \Omega \cdot \nabla\varphi - \varphi \cdot \nabla\Omega \ , \quad A\varphi = u \cdot \nabla\varphi - \varphi \cdot \nabla u \ ,$$

λ is a complex constant, and $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ is a complex 3-vector valued function.

This theorem was obtained by Steve Childress [5]. For a proof, see the proof of Proposition 1.

Remark 4.1 *This Lax pair has the great potential that it can be very useful in investigating finite time blow up solutions of 3D Euler equation, and in establishing global well-posedness of 3D Navier-Stokes equations.*

Remark 4.2 *When one does 2D reduction to the Lax pair (4.1), one gets $L = 0$. Therefore, the Lax pair (4.1) does not imply any Lax pair for 2D Euler equation.*

Remark 4.3 *The most promising research following from this Lax pair should be along the direction of Darboux transformations, group symmetries etc.. In building the inverse scattering transform, it is crucial to have constant coefficient differential operators in (especially the spatial part of) the Lax pair.*

Proposition 1 (Li,[13]) *We consider the following Lax pair,*

$$\begin{cases} L\varphi = \lambda\varphi \ , \\ \partial_t \varphi + A\varphi = 0 \ , \end{cases} \tag{4.2}$$

where

$$L\varphi = \Omega \cdot \nabla\varphi - \varphi \cdot \nabla\Omega + D_1\varphi \ , \quad A\varphi = q \cdot \nabla\varphi - \varphi \cdot \nabla q + D_2\varphi \ ,$$

λ is a complex constant, $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ is a complex 3-vector valued function, $q = (q_1, q_2, q_3)$ is a real 3-vector valued function, $D_j = \alpha^{(j)} \cdot \nabla$, $(j = 1, 2)$, and here $\alpha^{(j)} = (\alpha_1^{(j)}, \alpha_2^{(j)}, \alpha_3^{(j)})$ are real constant 3-vectors. The following equation instead of the 3D Euler equation,

$$\partial_t \Omega + (q \cdot \nabla)\Omega - (\Omega \cdot \nabla)q + D_2\Omega - D_1 q = 0 \ , \tag{4.3}$$

is a compatibility condition of this Lax pair. A specialization of (4.3) is the following system of equations,

$$\begin{cases} \partial_t \Omega + (q \cdot \nabla)\Omega - (\Omega \cdot \nabla)q = 0 \ , \\ D_1 q = D_2 \Omega \ . \end{cases}$$

A chance in building an inverse scattering transform [3] is that one can first build the inverse scattering transform for (4.2), and then take the limits

$$\alpha^{(j)} \to 0\ ,\ (j=1,2)\ ,\quad q \to u\ ,$$

to get results for 3D Euler equation.

Proof of Proposition 1: Following the same argument as in the proof of Theorem 3.1, one has the compatibility condition,

$$\Big[(\partial_t L) + [A, L]\Big]\varphi = 0\ .$$

$$(\partial_t L)\varphi = (\partial_t \Omega)\cdot\nabla\varphi - \varphi\cdot\nabla(\partial_t\Omega)\ .$$

$$\begin{aligned}
[A, L]\varphi &= q\cdot\nabla(\Omega\cdot\nabla\varphi - \varphi\cdot\nabla\Omega + D_1\varphi) \\
&\quad -(\Omega\cdot\nabla\varphi - \varphi\cdot\nabla\Omega + D_1\varphi)\cdot\nabla q \\
&\quad +D_2(\Omega\cdot\nabla\varphi - \varphi\cdot\nabla\Omega + D_1\varphi) \\
&\quad -\Omega\cdot\nabla(q\cdot\nabla\varphi - \varphi\cdot\nabla q + D_2\varphi) \\
&\quad +(q\cdot\nabla\varphi - \varphi\cdot\nabla q + D_2\varphi)\cdot\nabla\Omega \\
&\quad -D_1(q\cdot\nabla\varphi - \varphi\cdot\nabla q + D_2\varphi) \\
&= [(q\cdot\nabla)\Omega]\cdot\nabla\varphi + \underline{\Omega\cdot[(q\cdot\nabla)\nabla\varphi]}_{(1)} - \underline{[(q\cdot\nabla)\varphi]\cdot\nabla\Omega}_{(2)} \\
&\quad -\varphi\cdot[(q\cdot\nabla)\nabla\Omega] + \underline{(q\cdot\nabla)D_1\varphi}_{(3)} - \underline{[(\Omega\cdot\nabla)\varphi]\cdot\nabla q}_{(4)} \\
&\quad +[(\varphi\cdot\nabla)\Omega]\cdot\nabla q - \underline{(D_1\varphi)\cdot\nabla q}_{(5)} + (D_2\Omega)\cdot\nabla\varphi \\
&\quad +\underline{(\Omega\cdot\nabla)D_2\varphi}_{(6)} - \underline{(D_2\varphi)\cdot\nabla\Omega}_{(7)} - (\varphi\cdot\nabla)D_2\Omega \\
&\quad +\underline{D_2 D_1\varphi}_{(8)} - [(\Omega\cdot\nabla)q]\cdot\nabla\varphi - \underline{q\cdot[(\Omega\cdot\nabla)\nabla\varphi]}_{(1)} \\
&\quad +\underline{[(\Omega\cdot\nabla)\varphi]\cdot\nabla q}_{(4)} + \varphi\cdot[(\Omega\cdot\nabla)\nabla q] - \underline{(\Omega\cdot\nabla)D_2\varphi}_{(6)} \\
&\quad +\underline{[(q\cdot\nabla)\varphi]\cdot\nabla\Omega}_{(2)} - [(\varphi\cdot\nabla)q]\cdot\nabla\Omega + \underline{(D_2\varphi)\cdot\nabla\Omega}_{(7)} \\
&\quad -(D_1 q)\cdot\nabla\varphi - \underline{(q\cdot\nabla)D_1\varphi}_{(3)} + \underline{(D_1\varphi)\cdot\nabla q}_{(5)} \\
&\quad +(\varphi\cdot\nabla)D_1 q - \underline{D_1 D_2\varphi}_{(8)}\ .
\end{aligned}$$

All the terms $\underline{\quad}_{(j)}$, $1 \le j \le 8$, cancel each other. Thus, one has

$$\begin{aligned}
[A, L]\varphi &= \Big\{(q\cdot\nabla)\Omega - (\Omega\cdot\nabla)q - D_1 q + D_2\Omega\Big\}\cdot\nabla\varphi \\
&\quad -\varphi\cdot\nabla\Big\{(q\cdot\nabla)\Omega - (\Omega\cdot\nabla)q - D_1 q + D_2\Omega\Big\}\ .
\end{aligned}$$

Therefore the following equation

$$\partial_t\Omega + (q\cdot\nabla)\Omega - (\Omega\cdot\nabla)q + D_2\Omega - D_1 q = 0\ ,$$

is a compatibility condition. Since this equation involves two vaiables Ω and q, one can choose the specialization,

$$\begin{cases} \partial_t\Omega + (q\cdot\nabla)\Omega - (\Omega\cdot\nabla)q = 0\ , \\ D_1 q = D_2\Omega\ , \end{cases}$$

to be a compatibility condition. $\square$

5 A Darboux Transformation for 2D Euler Equation

Consider the Lax pair (3.1) at $\lambda = 0$, i.e.

$$\{\Omega, p\} = 0 \ , \tag{5.1}$$

$$\partial_t p + \{\Psi, p\} = 0 \ , \tag{5.2}$$

where we replaced the notation φ by p.

Theorem 5.1 *Let $f = f(t, x, y)$ be any fixed solution to the system (5.1, 5.2), we define the Gauge transform G_f:*

$$\tilde{p} = G_f p = \frac{1}{\Omega_x}[p_x - (\partial_x \ln f)p] \ , \tag{5.3}$$

and the transforms of the potentials Ω and Ψ:

$$\tilde{\Psi} = \Psi + F \ , \quad \tilde{\Omega} = \Omega + \Delta F \ , \tag{5.4}$$

where F is subject to the constraints

$$\{\Omega, \Delta F\} = 0 \ , \quad \{\Omega + \Delta F, F\} = 0 \ . \tag{5.5}$$

Then $\tilde{p}$ solves the system (5.1, 5.2) at $(\tilde{\Omega}, \tilde{\Psi})$. Thus (5.3) and (5.4) form the Darboux transformation for the 2D Euler equation (2.2) and its Lax pair (5.1, 5.2).

Remark 5.1 *For KdV equation and many other soliton equations, the Gauge transform is of the form [15],*

$$\tilde{p} = p_x - (\partial_x \ln f)p \ .$$

In general, Gauge transform does not involve potentials. For 2D Euler equation, a potential factor $\frac{1}{\Omega_x}$ is needed. From (5.1), one has

$$\frac{p_x}{\Omega_x} = \frac{p_y}{\Omega_y} \ .$$

The Gauge transform (5.3) can be rewritten as

$$\tilde{p} = \frac{p_x}{\Omega_x} - \frac{f_x}{\Omega_x}\frac{p}{f} = \frac{p_y}{\Omega_y} - \frac{f_y}{\Omega_y}\frac{p}{f} \ .$$

The Lax pair (5.1, 5.2) has a symmetry, i.e. it is invariant under the transform $(t, x, y) \to (-t, y, x)$. The form of the Gauge transform (5.3) resulted from the inclusion of the potential factor $\frac{1}{\Omega_x}$, is consistent with this symmetry.

Proposition 2 *If F satisfies the constraints*

$$\{\Omega, \Delta F\} = 0 \ , \quad \{\Omega, F\} = 0 \ , \tag{5.6}$$

or the constraints

$$\{\Omega, \Delta F\} = 0 \ , \quad \{\Delta F, F\} = 0 \ , \tag{5.7}$$

then F satisfies the constraints (5.5).

Proof: Notice that $\{\Omega, \Delta F\} = 0$ implies that $\Delta F_x = \frac{\Omega_x}{\Omega_y}\Delta F_y$. Thus,

$$
\begin{aligned}
\{\Omega + \Delta F, F\} &= (\Omega_x + \frac{\Omega_x}{\Omega_y}\Delta F_y)F_y - (\Omega_y + \Delta F_y)F_x \\
&= \frac{\Omega_x}{\Omega_y}(\Omega_y + \Delta F_y)F_y - (\Omega_y + \Delta F_y)F_x \\
&= \frac{(\Omega_y + \Delta F_y)}{\Omega_y}\{\Omega, F\} \ .
\end{aligned}
$$

Similarly, one has

$$\{\Omega + \Delta F, F\} = \frac{(\Omega_y + \Delta F_y)}{\Delta F_y}\{\Delta F, F\} \ .$$

Thus the claim in the proposition is true. □

Remark 5.2 *For soliton equations, F can be represented in terms of f [15].*

Proof of Theorem 5.1: One notices that using (5.1), (5.2) can be rewritten as

$$p_t = \frac{p_x}{\Omega_x}\{\Omega, \Psi\} \ . \tag{5.8}$$

To prove the theorem, one needs to check the two equations,

$$\{\tilde{\Omega}, \tilde{p}\} = 0 \ , \tag{5.9}$$

$$\tilde{p}_t = \frac{\tilde{p}_x}{\tilde{\Omega}_x}\{\tilde{\Omega}, \tilde{\Psi}\} \ . \tag{5.10}$$

First we check (5.9),

$$\{\tilde{\Omega}, \tilde{p}\} = \{\tilde{\Omega}, \frac{p_x f - p f_x}{\Omega_x f}\} = 0 \ ,$$

leads to

$$\frac{\tilde{\Omega}_x \Omega_y}{\Omega_x^2 f^2}A - \frac{\tilde{\Omega}_y \Omega_x}{\Omega_x^2 f^2}B = 0 \ , \tag{5.11}$$

where

$$
\begin{aligned}
A &= \frac{(p_{xy} f + p_x f_y - p_y f_x - p f_{xy})\Omega_x f - (p_x f - p f_x)(\Omega_{xy} f + \Omega_x f_y)}{\Omega_y} \ , \\
B &= \frac{(p_{xx} f - p f_{xx})\Omega_x f - (p_x f - p f_x)(\Omega_{xx} f + \Omega_x f_x)}{\Omega_x} \ .
\end{aligned}
$$

We will show that $A = B$. Using (5.1), one has

$$\left(\frac{p_y}{\Omega_y}\right)_x = \left(\frac{p_x}{\Omega_x}\right)_x \ ,$$

which leads to

$$\frac{p_{xy}}{\Omega_y} - \frac{p_y \Omega_{xy}}{\Omega_y^2} = \frac{p_{xx}}{\Omega_x} - \frac{p_x \Omega_{xx}}{\Omega_x^2} \ .$$

Using (5.1) again,

$$\frac{p_{xy}}{\Omega_y} - \frac{p_x\Omega_{xy}}{\Omega_x\Omega_y} = \frac{p_{xx}}{\Omega_x} - \frac{p_x\Omega_{xx}}{\Omega_x^2} \ .$$

Similarly,

$$\frac{f_{xy}}{\Omega_y} - \frac{f_x\Omega_{xy}}{\Omega_x\Omega_y} = \frac{f_{xx}}{\Omega_x} - \frac{f_x\Omega_{xx}}{\Omega_x^2} \ .$$

Thus,

$$\frac{p_{xy}f - pf_{xy}}{\Omega_y} - \frac{(p_xf - pf_x)\Omega_{xy}}{\Omega_x\Omega_y} = \frac{p_{xx}f - pf_{xx}}{\Omega_x} - \frac{(p_xf - pf_x)\Omega_{xx}}{\Omega_x^2} \ .$$

That is,

$$\frac{(p_{xy}f - pf_{xy})\Omega_x f}{\Omega_y} = \frac{(p_xf - pf_x)\Omega_{xy}f}{\Omega_y} + \frac{(p_{xx}f - pf_{xx})\Omega_x f}{\Omega_x} - \frac{(p_xf - pf_x)\Omega_{xx}f}{\Omega_x} \ . \tag{5.12}$$

Using (5.1) again, one has

$$\frac{p_xf_y - p_yf_x}{\Omega_y} = p_x\frac{f_x}{\Omega_x} - \frac{p_x}{\Omega_x}f_x = 0 \ . \tag{5.13}$$

Using (5.12), (5.13), and (5.1), we have

$$A = B = \frac{1}{\Omega_x}\Big[f^2\Omega_x p_{xx} - f(\Omega_x f)_x p_x + [f_x(\Omega_x f)_x - f_{xx}(\Omega_x f)]p\Big] \ .$$

Thus equation (5.11) becomes

$$\{\tilde{\Omega}, \Omega\}\frac{A}{\Omega_x^2 f^2} = 0 \ .$$

If we let

$$\{\Omega, \tilde{\Omega}\} = \{\Omega, \Delta F\} = 0 \ , \tag{5.14}$$

then (5.11) is satisfied.

Next we check (5.10).

$$\tilde{p}_t = \left(\frac{p_xf - pf_x}{\Omega_x f}\right)_t \ .$$

Using (5.8) for both p and f, one gets

$$\begin{aligned}
\tilde{p}_t &= \frac{1}{\Omega_x^2 f^2}\Big\{\Omega_x f\Big[f\Big(\frac{p_x}{\Omega_x}\{\Omega,\Psi\}\Big)_x + p_x\Big(\frac{f_x}{\Omega_x}\{\Omega,\Psi\}\Big) \\
&\quad - f_x\Big(\frac{p_x}{\Omega_x}\{\Omega,\Psi\}\Big) - p\Big(\frac{f_x}{\Omega_x}\{\Omega,\Psi\}\Big)_x\Big] \\
&\quad - (p_xf - pf_x)\Big[f\{\Omega,\Psi\}_x + \Omega_x\frac{f_x}{\Omega_x}\{\Omega,\Psi\}\Big]\Big\} \\
&= (fp_{xx} - pf_{xx})\frac{\{\Omega,\Psi\}}{\Omega_x^2 f} \\
&\quad + (fp_x - pf_x)\Big[\frac{1}{\Omega_x f}\Big(\frac{\{\Omega,\Psi\}}{\Omega_x}\Big)_x - \frac{1}{\Omega_x^2 f^2}\Big(f\{\Omega,\Psi\}\Big)_x\Big] \ .
\end{aligned} \tag{5.15}$$

The right hand side of (5.10) is,

$$\begin{aligned}\frac{\tilde{p}_x}{\tilde{\Omega}_x}\{\tilde{\Omega},\tilde{\Psi}\} &= (fp_{xx}-pf_{xx})\frac{1}{\Omega_x f}\frac{\{\Omega+\Delta F,\Psi+F\}}{\Omega_x+\Delta F_x} \\ &\quad +(fp_x-pf_x)\frac{-(\Omega_x f)_x}{\Omega_x^2 f^2}\frac{\{\Omega+\Delta F,\Psi+F\}}{\Omega_x+\Delta F_x}\ .\end{aligned}\tag{5.16}$$

From (5.15) and (5.16), the coefficients of "$fp_{xx}-pf_{xx}$" leads to

$$\frac{1}{\Omega_x f}\frac{\{\Omega,\Psi\}}{\Omega_x}=\frac{1}{\Omega_x f}\frac{\{\Omega+\Delta F,\Psi+F\}}{\Omega_x+\Delta F_x}\ ,\tag{5.17}$$

and the coefficients of "fp_x-pf_x" leads to

$$\frac{1}{\Omega_x f}\Big(\frac{\{\Omega,\Psi\}}{\Omega_x}\Big)_x-\frac{1}{\Omega_x^2 f^2}\Big(f\{\Omega,\Psi\}\Big)_x=\frac{-(\Omega_x f)_x}{\Omega_x^2 f^2}\frac{\{\Omega+\Delta F,\Psi+F\}}{\Omega_x+\Delta F_x}\ .\tag{5.18}$$

We will show that (5.17) implies (5.18). The left hand side of (5.18) is

$$\begin{aligned}&\frac{1}{\Omega_x f}\Big(\frac{\{\Omega,\Psi\}}{\Omega_x}\Big)_x-\frac{1}{\Omega_x^2 f^2}\Big(\frac{\Omega_x f\{\Omega,\Psi\}}{\Omega_x}\Big)_x \\ =\ &\frac{1}{\Omega_x f}\Big(\frac{\{\Omega,\Psi\}}{\Omega_x}\Big)_x-\frac{1}{\Omega_x^2 f^2}\Omega_x f\Big(\frac{\{\Omega,\Psi\}}{\Omega_x}\Big)_x \\ &-\frac{1}{\Omega_x^2 f^2}(\Omega_x f)_x\frac{\{\Omega,\Psi\}}{\Omega_x}=-\frac{1}{\Omega_x^2 f^2}(\Omega_x f)_x\frac{\{\Omega,\Psi\}}{\Omega_x}\ .\end{aligned}$$

Thus (5.18) becomes

$$-\frac{1}{\Omega_x^2 f^2}(\Omega_x f)_x\frac{\{\Omega,\Psi\}}{\Omega_x}=\frac{-(\Omega_x f)_x}{\Omega_x^2 f^2}\frac{\{\Omega+\Delta F,\Psi+F\}}{\Omega_x+\Delta F_x}\ .$$

Therefore, (5.17) implies (5.18). From (5.17), we have

$$(\Omega_x+\Delta F_x)\{\Omega,\Psi\}=\Omega_x\{\Omega+\Delta F,\Psi+F\}\ ,$$

which leads to

$$\Delta F_x\{\Omega,\Psi\}=\Omega_x[\{\Omega,F\}+\{\Delta F,F\}+\{\Delta F,\Psi\}]\ .\tag{5.19}$$

Notice that

$$\Delta F_x\{\Omega,\Psi\}-\Omega_x\{\Delta F,\Psi\}=-\Psi_x\{\Omega,\Delta F\}=0\ ,$$

by (5.14). Then (5.19) becomes

$$\Omega_x[\{\Omega,F\}+\{\Delta F,F\}]=0\ .\tag{5.20}$$

Thus if

$$\{\Omega+\Delta F,F\}=0\ ,$$

(5.20) is fulfilled. □

6 Conclusion

We have briefly reported the most recent results on Lax pairs and a Darboux transformation for Euler equations of incompressible inviscid fluid flows. The most promising researches following from these Lax pairs should be along the directions of Darboux transformations, group symmetries, etc. These Lax pairs have the great potentials that they can be very useful in investigating finite time blow up solutions of 3D Euler equations, and in establishing global well-posedness of 3D Navier-Stokes equations.

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